6.3 Separation of Variables and the Logistic Equation

- **Recognize and solve differential equations that can be solved by separation of variables.**
- **Use differential equations to model and solve applied problems.**
- **Solve and analyze logistic differential equations.**

Separation of Variables

Consider a differential equation that can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

where M is a continuous function of x alone and N is a continuous function of y alone. As you saw in Section 6.2, for this type of equation, all x terms can be collected with dx and all y terms with dy, and a solution can be obtained by integration. Such equations are said to be **separable,** and the solution procedure is called **separation of variables.** Below are some examples of differential equations that are separable.

Original Differential Equation	Rewritten with Variables Separated
$x^2 + 3y\frac{dy}{dx} = 0$	$3y\,dy = -x^2\,dx$
$(\sin x)y' = \cos x$	$dy = \cot x\,dx$
$\frac{xy'}{e^y + 1} = 2$	$\frac{1}{e^y + 1}\,dy = \frac{2}{x}\,dx$

EXAMPLE 1 Separation of Variables

See LarsonCalculus.com for an interactive version of this type of example.

Find the general solution of

$$(x^2 + 4)\frac{dy}{dx} = xy.$$

Solution To begin, note that $y = 0$ is a solution. To find other solutions, assume that $y \neq 0$ and separate variables as shown.

$$(x^2 + 4)\,dy = xy\,dx \qquad \text{Differential form}$$

$$\frac{dy}{y} = \frac{x}{x^2 + 4}\,dx \qquad \text{Separate variables.}$$

Now, integrate to obtain

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 4}\,dx \qquad \text{Integrate.}$$

$$\ln|y| = \frac{1}{2}\ln(x^2 + 4) + C_1$$

$$\ln|y| = \ln\sqrt{x^2 + 4} + C_1$$

$$|y| = e^{C_1}\sqrt{x^2 + 4}$$

$$y = \pm e^{C_1}\sqrt{x^2 + 4}.$$

Because $y = 0$ is also a solution, you can write the general solution as

$$y = C\sqrt{x^2 + 4}. \qquad \text{General solution}$$

REMARK Be sure to check your solutions throughout this chapter. In Example 1, you can check the solution

$$y = C\sqrt{x^2 + 4}$$

by differentiating and substituting into the original equation.

$$(x^2 + 4)\frac{dy}{dx} = xy$$

$$(x^2 + 4)\frac{Cx}{\sqrt{x^2 + 4}} \stackrel{?}{=} x\left(C\sqrt{x^2 + 4}\right)$$

$$Cx\sqrt{x^2 + 4} = Cx\sqrt{x^2 + 4}$$

So, the solution checks.

In some cases, it is not feasible to write the general solution in the explicit form $y = f(x)$. The next example illustrates such a solution. Implicit differentiation can be used to verify this solution.

FOR FURTHER INFORMATION For an example (from engineering) of a differential equation that is separable, see the article "Designing a Rose Cutter" by J. S. Hartzler in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

EXAMPLE 2 Finding a Particular Solution

Given the initial condition $y(0) = 1$, find the particular solution of the equation

$$xy\,dx + e^{-x^2}(y^2 - 1)\,dy = 0.$$

Solution Note that $y = 0$ is a solution of the differential equation—but this solution does not satisfy the initial condition. So, you can assume that $y \neq 0$. To separate variables, you must rid the first term of y and the second term of e^{-x^2}. So, you should multiply by e^{x^2}/y and obtain the following.

$$xy\,dx + e^{-x^2}(y^2 - 1)\,dy = 0$$
$$e^{-x^2}(y^2 - 1)\,dy = -xy\,dx$$
$$\int\left(y - \frac{1}{y}\right)dy = \int -xe^{x^2}\,dx$$
$$\frac{y^2}{2} - \ln|y| = -\frac{1}{2}e^{x^2} + C$$

From the initial condition $y(0) = 1$, you have

$$\frac{1}{2} - 0 = -\frac{1}{2} + C$$

which implies that $C = 1$. So, the particular solution has the implicit form

$$\frac{y^2}{2} - \ln|y| = -\frac{1}{2}e^{x^2} + 1$$
$$y^2 - \ln y^2 + e^{x^2} = 2.$$

You can check this by differentiating and rewriting to get the original equation.

EXAMPLE 3 Finding a Particular Solution Curve

Find the equation of the curve that passes through the point $(1, 3)$ and has a slope of y/x^2 at any point (x, y).

Solution Because the slope of the curve is y/x^2, you have

$$\frac{dy}{dx} = \frac{y}{x^2}$$

with the initial condition $y(1) = 3$. Separating variables and integrating produces

$$\int \frac{dy}{y} = \int \frac{dx}{x^2}, \quad y \neq 0$$
$$\ln|y| = -\frac{1}{x} + C_1$$
$$y = e^{-(1/x) + C_1}$$
$$y = Ce^{-1/x}.$$

Because $y = 3$ when $x = 1$, it follows that $3 = Ce^{-1}$ and $C = 3e$. So, the equation of the specified curve is

$$y = (3e)e^{-1/x} \quad \Rightarrow \quad y = 3e^{(x-1)/x}, \quad x > 0.$$

Because the solution is not defined at $x = 0$ and the initial condition is given at $x = 1$, x is restricted to positive values. See Figure 6.11.

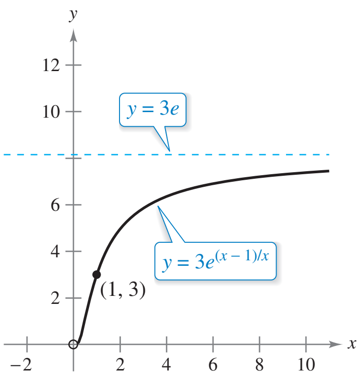


Figure 6.11

Applications

EXAMPLE 4 Wildlife Population

The rate of change of the number of coyotes $N(t)$ in a population is directly proportional to $650 - N(t)$, where t is the time in years. When $t = 0$, the population is 300, and when $t = 2$, the population has increased to 500. Find the population when $t = 3$.

Solution Because the rate of change of the population is proportional to $650 - N(t)$, or $650 - N$, you can write the differential equation

$$\frac{dN}{dt} = k(650 - N).$$

You can solve this equation using separation of variables.

$$\begin{aligned}
dN &= k(650 - N)\,dt && \text{Differential form} \\
\frac{dN}{650 - N} &= k\,dt && \text{Separate variables.} \\
-\ln|650 - N| &= kt + C_1 && \text{Integrate.} \\
\ln|650 - N| &= -kt - C_1 && \\
650 - N &= e^{-kt - C_1} && \text{Assume } N < 650. \\
N &= 650 - Ce^{-kt} && \text{General solution}
\end{aligned}$$

Using $N = 300$ when $t = 0$, you can conclude that $C = 350$, which produces

$$N = 650 - 350e^{-kt}.$$

Then, using $N = 500$ when $t = 2$, it follows that

$$500 = 650 - 350e^{-2k} \quad \Rightarrow \quad e^{-2k} = \frac{3}{7} \quad \Rightarrow \quad k \approx 0.4236.$$

So, the model for the coyote population is

$$N = 650 - 350e^{-0.4236t}. \qquad \text{Model for population}$$

When $t = 3$, you can approximate the population to be

$$\begin{aligned}
N &= 650 - 350e^{-0.4236(3)} \\
&\approx 552 \text{ coyotes.}
\end{aligned}$$

The model for the population is shown in Figure 6.12. Note that $N = 650$ is the horizontal asymptote of the graph and is the *carrying capacity* of the model. You will learn more about carrying capacity later in this section.

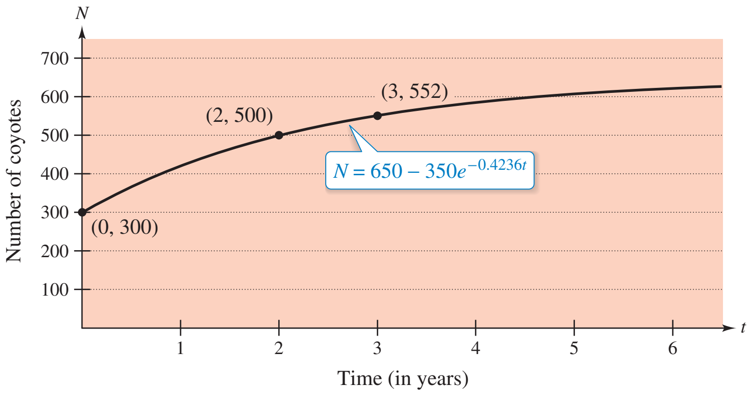


Figure 6.12

franzfoto.com/Alamy

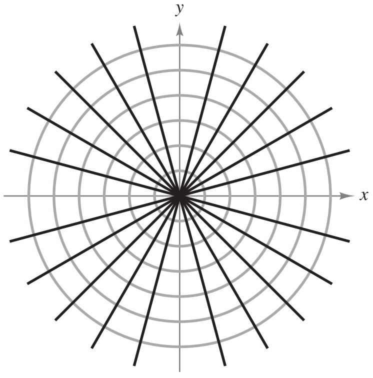
A common problem in electrostatics, thermodynamics, and hydrodynamics involves finding a family of curves, each of which is orthogonal to all members of a given family of curves. For example, Figure 6.13 shows a family of circles

$$x^2 + y^2 = C \quad \text{Family of circles}$$

each of which intersects the lines in the family

$$y = Kx \quad \text{Family of lines}$$

at right angles. Two such families of curves are said to be **mutually orthogonal,** and each curve in one of the families is called an **orthogonal trajectory** of the other family. In electrostatics, lines of force are orthogonal to the *equipotential curves*. In thermodynamics, the flow of heat across a plane surface is orthogonal to the *isothermal curves*. In hydrodynamics, the flow (stream) lines are orthogonal trajectories of the *velocity potential curves*.



Each line $y = Kx$ is an orthogonal trajectory of the family of circles.
Figure 6.13

EXAMPLE 5 Finding Orthogonal Trajectories

Describe the orthogonal trajectories for the family of curves given by

$$y = \frac{C}{x}$$

for $C \neq 0$. Sketch several members of each family.

Solution First, solve the given equation for C and write $xy = C$. Then, by differentiating implicitly with respect to x, you obtain the differential equation

$$x\frac{dy}{dx} + y = 0 \quad \text{Differential equation}$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}. \quad \text{Slope of given family}$$

Because dy/dx represents the slope of the given family of curves at (x, y), it follows that the orthogonal family has the negative reciprocal slope x/y. So,

$$\frac{dy}{dx} = \frac{x}{y}. \quad \text{Slope of orthogonal family}$$

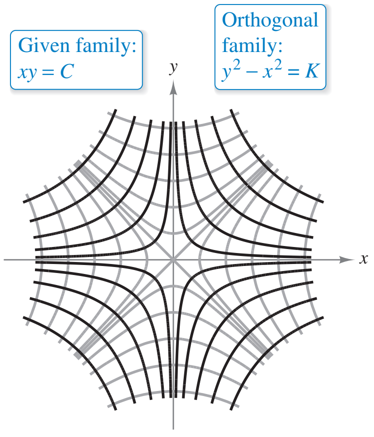
Now you can find the orthogonal family by separating variables and integrating.

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = K$$

The centers are at the origin, and the transverse axes are vertical for $K > 0$ and horizontal for $K < 0$. When $K = 0$, the orthogonal trajectories are the lines $y = \pm x$. When $K \neq 0$, the orthogonal trajectories are hyperbolas. Several trajectories are shown in Figure 6.14.



Orthogonal trajectories
Figure 6.14

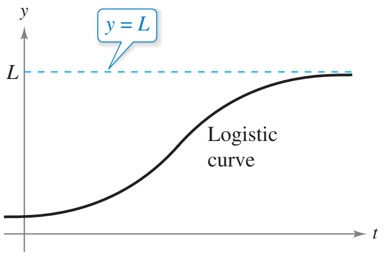
Logistic Differential Equation

In Section 6.2, the exponential growth model was derived from the fact that the rate of change of a variable y is proportional to the value of y. You observed that the differential equation $dy/dt = ky$ has the general solution $y = Ce^{kt}$. Exponential growth is unlimited, but when describing a population, there often exists some upper limit L past which growth cannot occur. This upper limit L is called the **carrying capacity,** which is the maximum population $y(t)$ that can be sustained or supported as time t increases. A model that is often used to describe this type of growth is the **logistic differential equation**

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Logistic differential equation}$$

where k and L are positive constants. A population that satisfies this equation does not grow without bound, but approaches the carrying capacity L as t increases.

From the equation, you can see that if y is between 0 and the carrying capacity L, then $dy/dt > 0$, and the population increases. If y is greater than L, then $dy/dt < 0$, and the population decreases. The graph of the function y is called the *logistic curve*, as shown in Figure 6.15.



Note that as $t \to \infty$, $y \to L$.
Figure 6.15

EXAMPLE 6 Deriving the General Solution

Solve the logistic differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right).$$

Solution Begin by separating variables.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Write differential equation.}$$

$$\frac{1}{y(1 - y/L)}\,dy = k\,dt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y(1 - y/L)}\,dy = \int k\,dt \qquad \text{Integrate each side.}$$

$$\int \left(\frac{1}{y} + \frac{1}{L - y}\right)dy = \int k\,dt \qquad \text{Rewrite left side using partial fractions.}$$

$$\ln|y| - \ln|L - y| = kt + C \qquad \text{Find antiderivative of each side.}$$

$$\ln\left|\frac{L - y}{y}\right| = -kt - C \qquad \text{Multiply each side by } -1 \text{ and simplify.}$$

$$\left|\frac{L - y}{y}\right| = e^{-kt - C} \qquad \text{Exponentiate each side.}$$

$$\left|\frac{L - y}{y}\right| = e^{-C}e^{-kt} \qquad \text{Property of exponents}$$

$$\frac{L - y}{y} = be^{-kt} \qquad \text{Let } \pm e^{-C} = b.$$

Solving this equation for y produces $y = \dfrac{L}{1 + be^{-kt}}$. ■

• • REMARK A review of the method of partial fractions is given in Section 8.5.

From Example 6, you can conclude that all solutions of the logistic differential equation are of the general form

$$y = \frac{L}{1 + be^{-kt}}.$$

Exploration

Use a graphing utility to investigate the effects of the values of L, b, and k on the graph of

$$y = \frac{L}{1 + be^{-kt}}.$$

Include some examples to support your results.

EXAMPLE 7 Solving a Logistic Differential Equation

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{4000}\right), \quad 40 \le p \le 4000$$

where t is the number of years.

a. Write a model for the elk population in terms of t.

b. Graph the slope field for the differential equation and the solution that passes through the point $(0, 40)$.

c. Use the model to estimate the elk population after 15 years.

d. Find the limit of the model as $t \to \infty$.

Solution

a. You know that $L = 4000$. So, the solution of the equation is of the form

$$p = \frac{4000}{1 + be^{-kt}}.$$

Because $p(0) = 40$, you can solve for b as follows.

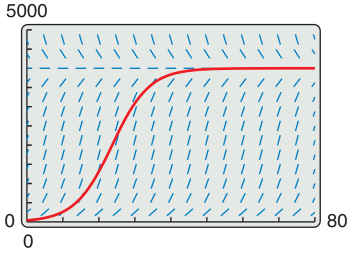
$$40 = \frac{4000}{1 + be^{-k(0)}} \quad \Rightarrow \quad 40 = \frac{4000}{1 + b} \quad \Rightarrow \quad b = 99$$

Then, because $p = 104$ when $t = 5$, you can solve for k.

$$104 = \frac{4000}{1 + 99e^{-k(5)}} \quad \Rightarrow \quad k \approx 0.194$$

So, a model for the elk population is

$$p = \frac{4000}{1 + 99e^{-0.194t}}.$$



Slope field for

$$\frac{dp}{dt} = 0.194p\left(1 - \frac{p}{4000}\right)$$

and the solution passing through $(0, 40)$

Figure 6.16

b. Using a graphing utility, you can graph the slope field for

$$\frac{dp}{dt} = 0.194p\left(1 - \frac{p}{4000}\right)$$

and the solution that passes through $(0, 40)$, as shown in Figure 6.16.

c. To estimate the elk population after 15 years, substitute 15 for t in the model.

$$p = \frac{4000}{1 + 99e^{-0.194(15)}} \qquad \text{Substitute 15 for } t.$$

$$= \frac{4000}{1 + 99e^{-2.91}} \qquad \text{Simplify.}$$

$$\approx 626$$

d. As t increases without bound, the denominator of

$$\frac{4000}{1 + 99e^{-0.194t}}$$

gets closer and closer to 1. So,

$$\lim_{t \to \infty} \frac{4000}{1 + 99e^{-0.194t}} = 4000.$$

6.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a General Solution Using Separation of Variables In Exercises 1–14, find the general solution of the differential equation.

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$
2. $\dfrac{dy}{dx} = \dfrac{3x^2}{y^2}$
3. $x^2 + 5y\dfrac{dy}{dx} = 0$
4. $\dfrac{dy}{dx} = \dfrac{6 - x^2}{2y^3}$
5. $\dfrac{dr}{ds} = 0.75r$
6. $\dfrac{dr}{ds} = 0.75s$
7. $(2 + x)y' = 3y$
8. $xy' = y$
9. $yy' = 4 \sin x$
10. $yy' = -8 \cos(\pi x)$
11. $\sqrt{1 - 4x^2}\, y' = x$
12. $\sqrt{x^2 - 16}\, y' = 11x$
13. $y \ln x - xy' = 0$
14. $12yy' - 7e^x = 0$

Finding a Particular Solution Using Separation of Variables In Exercises 15–24, find the particular solution that satisfies the initial condition.

	Differential Equation	Initial Condition
15.	$yy' - 2e^x = 0$	$y(0) = 3$
16.	$\sqrt{x} + \sqrt{y}\, y' = 0$	$y(1) = 9$
17.	$y(x + 1) + y' = 0$	$y(-2) = 1$
18.	$2xy' - \ln x^2 = 0$	$y(1) = 2$
19.	$y(1 + x^2)y' - x(1 + y^2) = 0$	$y(0) = \sqrt{3}$
20.	$y\sqrt{1 - x^2}\, y' - x\sqrt{1 - y^2} = 0$	$y(0) = 1$
21.	$\dfrac{du}{dv} = uv \sin v^2$	$u(0) = 1$
22.	$\dfrac{dr}{ds} = e^{r - 2s}$	$r(0) = 0$
23.	$dP - kP\, dt = 0$	$P(0) = P_0$
24.	$dT + k(T - 70)\, dt = 0$	$T(0) = 140$

Finding a Particular Solution In Exercises 25–28, find an equation of the graph that passes through the point and has the given slope.

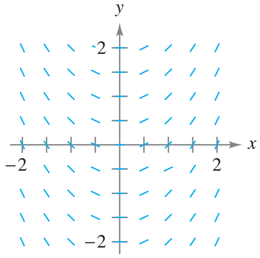
25. $(0, 2)$, $y' = \dfrac{x}{4y}$
26. $(1, 1)$, $y' = -\dfrac{9x}{16y}$
27. $(9, 1)$, $y' = \dfrac{y}{2x}$
28. $(8, 2)$, $y' = \dfrac{2y}{3x}$

Using Slope In Exercises 29 and 30, find all functions f having the indicated property.

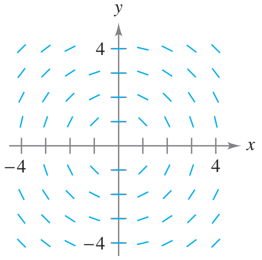
29. The tangent to the graph of f at the point (x, y) intersects the x-axis at $(x + 2, 0)$.
30. All tangents to the graph of f pass through the origin.

Slope Field In Exercises 31 and 32, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to *MathGraphs.com*.

31. $\dfrac{dy}{dx} = x$

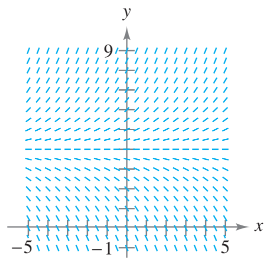


32. $\dfrac{dy}{dx} = -\dfrac{x}{y}$

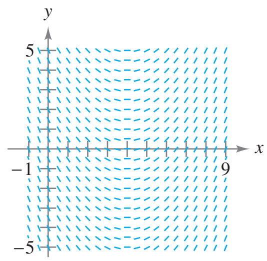


Slope Field In Exercises 33–36, (a) write a differential equation for the statement, (b) match the differential equation with a possible slope field, and (c) verify your result by using a graphing utility to graph a slope field for the differential equation. [The slope fields are labeled (a), (b), (c), and (d).] To print an enlarged copy of the graph, go to *MathGraphs.com*.

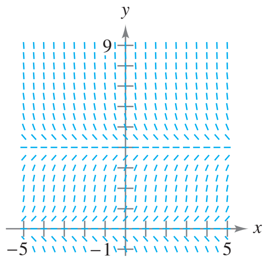
(a)



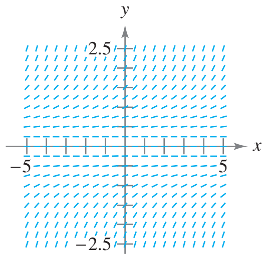
(b)



(c)



(d)



33. The rate of change of y with respect to x is proportional to the difference between y and 4.
34. The rate of change of y with respect to x is proportional to the difference between x and 4.
35. The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4.
36. The rate of change of y with respect to x is proportional to y^2.
37. **Radioactive Decay** The rate of decomposition of radioactive radium is proportional to the amount present at any time. The half-life of radioactive radium is 1599 years. What percent of a present amount will remain after 50 years?

38. Chemical Reaction In a chemical reaction, a certain compound changes into another compound at a rate proportional to the unchanged amount. There is 40 grams of the original compound initially and 35 grams after 1 hour. When will 75 percent of the compound be changed?

39. Weight Gain A calf that weighs 60 pounds at birth gains weight at the rate

$$\frac{dw}{dt} = k(1200 - w)$$

where w is weight in pounds and t is time in years.

(a) Solve the differential equation.

(b) Use a graphing utility to graph the particular solutions for $k = 0.8, 0.9$, and 1.

(c) The animal is sold when its weight reaches 800 pounds. Find the time of sale for each of the models in part (b).

(d) What is the maximum weight of the animal for each of the models in part (b)?

40. Weight Gain A calf that weighs w_0 pounds at birth gains weight at the rate $dw/dt = 1200 - w$, where w is weight in pounds and t is time in years. Solve the differential equation.

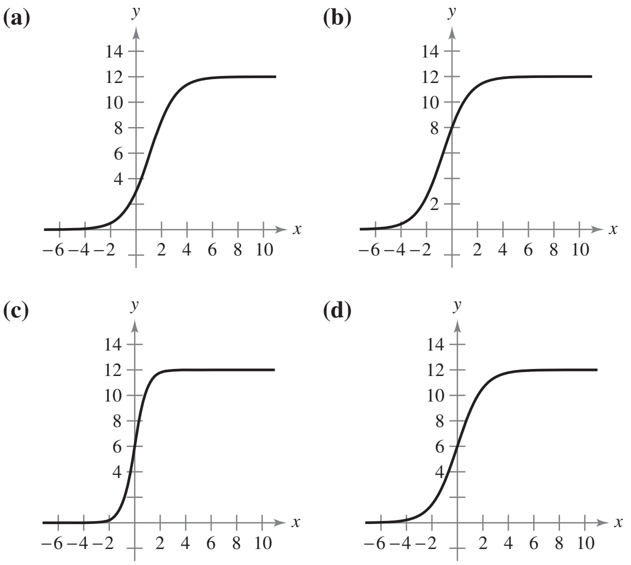
Finding Orthogonal Trajectories **In Exercises 41–46, find the orthogonal trajectories of the family. Use a graphing utility to graph several members of each family.**

41. $x^2 + y^2 = C$ **42.** $x^2 - 2y^2 = C$

43. $x^2 = Cy$ **44.** $y^2 = 2Cx$

45. $y^2 = Cx^3$ **46.** $y = Ce^x$

Matching **In Exercises 47–50, match the logistic equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**



47. $y = \dfrac{12}{1 + e^{-x}}$ **48.** $y = \dfrac{12}{1 + 3e^{-x}}$

49. $y = \dfrac{12}{1 + \frac{1}{2}e^{-x}}$ **50.** $y = \dfrac{12}{1 + e^{-2x}}$

Using a Logistic Equation **In Exercises 51 and 52, the logistic equation models the growth of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.**

51. $P(t) = \dfrac{2100}{1 + 29e^{-0.75t}}$ **52.** $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

Using a Logistic Differential Equation **In Exercises 53 and 54, the logistic differential equation models the growth rate of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) graph a slope field using a computer algebra system, and (d) determine the value of P at which the population growth rate is the greatest.**

53. $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$ **54.** $\dfrac{dP}{dt} = 0.1P - 0.0004P^2$

Solving a Logistic Differential Equation **In Exercises 55–58, find the logistic equation that passes through the given point.**

55. $\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{36}\right)$, $(0, 4)$ **56.** $\dfrac{dy}{dt} = 2.8y\left(1 - \dfrac{y}{10}\right)$, $(0, 7)$

57. $\dfrac{dy}{dt} = \dfrac{4y}{5} - \dfrac{y^2}{150}$, $(0, 8)$ **58.** $\dfrac{dy}{dt} = \dfrac{3y}{20} - \dfrac{y^2}{1600}$, $(0, 15)$

59. Endangered Species A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

(a) Write a logistic equation that models the population of panthers in the preserve.

(b) Find the population after 5 years.

(c) When will the population reach 100?

(d) Write a logistic differential equation that models the growth rate of the panther population. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answer.

(e) At what time is the panther population growing most rapidly? Explain.

60. Bacteria Growth At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 4 grams. The maximum weight of the culture is 20 grams.

(a) Write a logistic equation that models the weight of the bacterial culture.

(b) Find the culture's weight after 5 hours.

(c) When will the culture's weight reach 18 grams?

(d) Write a logistic differential equation that models the growth rate of the culture's weight. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answer.

(e) At what time is the culture's weight increasing most rapidly? Explain.

WRITING ABOUT CONCEPTS

61. Separation of Variables In your own words, describe how to recognize and solve differential equations that can be solved by separation of variables.

62. Mutually Orthogonal In your own words, describe the relationship between two families of curves that are mutually orthogonal.

63. Finding a Derivative Show that if

$$y = \frac{1}{1 + be^{-kt}}$$

then

$$\frac{dy}{dt} = ky(1 - y).$$

64. Point of Inflection For any logistic growth curve, show that the point of inflection occurs at $y = L/2$ when the solution starts below the carrying capacity L.

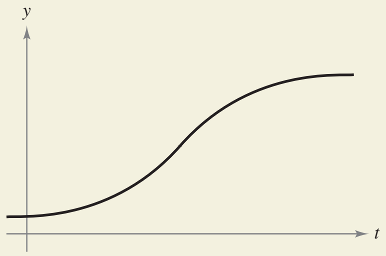
65. Sailing

Ignoring resistance, a sailboat starting from rest accelerates (dv/dt) at a rate proportional to the difference between the velocities of the wind and the boat.

(a) The wind is blowing at 20 knots, and after 1 half-hour, the boat is moving at 10 knots. Write the velocity v as a function of time t.

(b) Use the result of part (a) to write the distance traveled by the boat as a function of time.

66. HOW DO YOU SEE IT? The growth of a population is modeled by a logistic equation as shown in the graph below. What happens to the rate of growth as the population increases? What do you think causes this to occur in real-life situations, such as animal or human populations?



Determining if a Function Is Homogeneous **In Exercises 67–74, determine whether the function is homogeneous, and if it is, determine its degree. A function $f(x, y)$ is *homogeneous of degree n* if $f(tx, ty) = t^n f(x, y)$.**

67. $f(x, y) = x^3 - 4xy^2 + y^3$

68. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

69. $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$

70. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

71. $f(x, y) = 2 \ln xy$

72. $f(x, y) = \tan(x + y)$

73. $f(x, y) = 2 \ln \dfrac{x}{y}$

74. $f(x, y) = \tan \dfrac{y}{x}$

Solving a Homogeneous Differential Equation **In Exercises 75–80, solve the homogeneous differential equation in terms of x and y. A *homogeneous differential equation* is an equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$, where M and N are homogeneous functions of the same degree. To solve an equation of this form by the method of separation of variables, use the substitutions $y = vx$ and $dy = x\,dv + v\,dx$.**

75. $(x + y)\,dx - 2x\,dy = 0$

76. $(x^3 + y^3)\,dx - xy^2\,dy = 0$

77. $(x - y)\,dx - (x + y)\,dy = 0$

78. $(x^2 + y^2)\,dx - 2xy\,dy = 0$

79. $xy\,dx + (y^2 - x^2)\,dy = 0$

80. $(2x + 3y)\,dx - x\,dy = 0$

True or False? **In Exercises 81–83, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

81. The function $y = 0$ is always a solution of a differential equation that can be solved by separation of variables.

82. The differential equation $y' = xy - 2y + x - 2$ can be written in separated variables form.

83. The families $x^2 + y^2 = 2Cy$ and $x^2 + y^2 = 2Kx$ are mutually orthogonal.

PUTNAM EXAM CHALLENGE

84. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

This problem was composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

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